Math 250 6.4 Volume: The Shell Method

Objectives 1) Find the volume of a solid of revolution using the cylindrical shell method.

2) Compare the shell method to the disk method

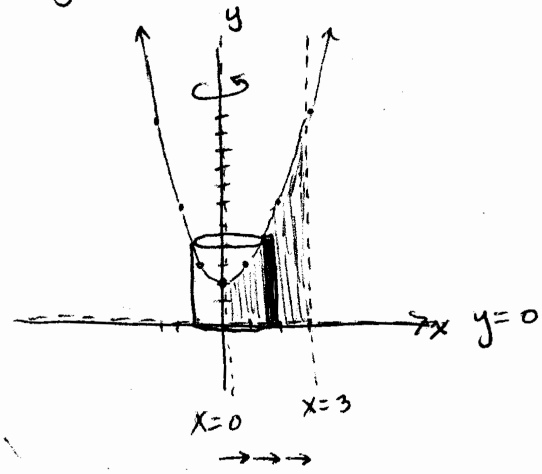
Primary concept: definite integral is accumulating the surface areas of cylinders whose radii (radius plural) are expanding along the axis of integral

Notice:

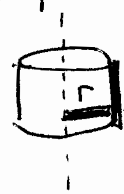
- Around what line do I revolve?
- Do I integrate in x or in y?
- Have I written my integrand using the variable with which I should integrate?
- Have I written my limits of integration with the correct variable?

Math 250

① Revolve $f(x) = x^2+2$, bounded by $x=0$ and $x=3$, $y=0$ around the y-axis. Find the volume of this region of revolution using the method of cylindrical shells.



sample cylinder



h = vertical bar
r = horizontal bar

axis of revolution passes through center of all cylinders as we integrate.

Integrate shells as radius increases ⇒ $x=0$ to $x=3$ } in x

r(x) is just x! $r \approx 0$ ⟶ $r=3$

As radius increases, height varies with parabola } express height in x also.

$h(x) = x^2+2$

top − bottom
$(x^2+2) - (0)$

$$\text{Volume} = \int_{x=0}^{x=3} 2\pi\, r(x)\, h(x)\, dx$$

$$= \int_{x=0}^{3} 2\pi\,(x)(x^2+2-0)\,dx$$ subst $r(x) = x$, $h(x) = x^2+2-0$

$$= 2\pi \int_0^3 x(x^2+2)\,dx$$ simplify

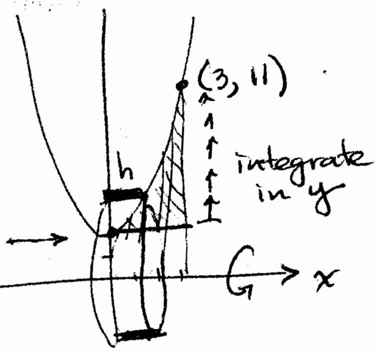
$$= 2\pi \int_0^3 x^3+2x\,dx$$ distribute

$$= 2\pi \left[\tfrac{1}{4}x^4 + x^2\right]\Big|_0^3$$ antidiff

$$= 2\pi\left[\left(\tfrac{1}{4}\cdot 3^4 + 3^2\right) - 0\right] = \boxed{\tfrac{117}{2}\pi}$$

Math 250

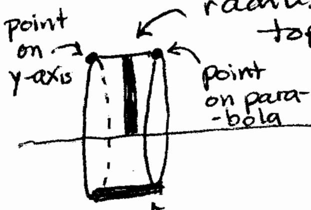
(2) Revolve the region bounded by $y=x^2+2$, $x=0$, $x=3$, $y=2$ around the x-axis and find the volume of revolution using the method of cylindrical shells.



Smallest cylinder starts here

similar region to previous question, but revolved around different axis

Sample cylinder



radius is a vertical bar
top y − bottom y

axis of revolution passes through center of sample cylinder

height is a horizontal bar
right x − left x ⇒ easier seen at top of cylinder!

(parabola x-coord) − (y-axis x-coord)

For shells,
Integrate in radius
from $y=2$ to $y=11$

Need x coordinate in terms of y.

Solve $y=x^2+2$ for x

$$y-2=x^2$$

$$\pm\sqrt{y-2}=x$$

want QI ⇒ $+\sqrt{\ }$

$$x=\sqrt{y-2}$$

$$r(y)=y$$

$$h(y)=\sqrt{y-2}-0=\sqrt{y-2}$$

$$\text{Volume}=\int_{y=2}^{11}2\pi\, r(y)\,h(y)\,dy$$

$$=\int_2^{11}2\pi(y)(\sqrt{y-2})\,dy$$

$u+2=y \rightarrow u=y-2$
$du=dy$

$u_1=2-2=0$
$u_2=11-2=9$

need u-substitution!

$$= \int_{u=0}^{u=9} 2\pi (u+2) u^{1/2} \, du$$

rewrite in u and du
u-substitution

$$= 2\pi \int_{u=0}^{9} u^{1/2} (u+2) \, du$$

rearrange

→ constant multiple outside integral
→ $u^{1/2}$ to front, distribu

$$= 2\pi \int_{u=0}^{9} u^{3/2} + 2u^{1/2} \, du$$

dist

$$= 2\pi \left[\frac{2}{5} u^{5/2} + 2 \cdot \frac{2}{3} u^{3/2} \right] \Big|_0^9$$

antidiff

$$= 2\pi \left[\frac{2}{5} u^{5/2} + \frac{4}{3} u^{3/2} \right] \Big|_0^9$$

simplify

<u>option 1</u>: Evaluate in u.

$$= 2\pi \left[\frac{2}{5}(9)^{5/2} + \frac{4}{3}(9)^{3/2} - 0 \right]$$

$$= 2\pi \left[\frac{2}{5}(3)^5 + \frac{4}{3}(3)^3 \right]$$

$$= \boxed{\frac{1332}{5}\pi}$$

<u>Option 2</u>: Evaluate in y.

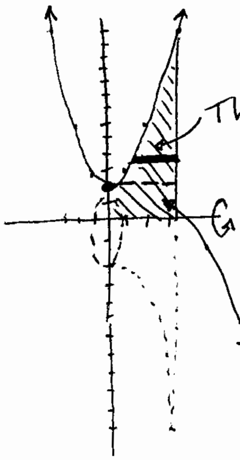
$$= 2\pi \left[\frac{2}{5}(y-2)^{5/2} + \frac{4}{3}(y-2)^{3/2} \right] \Big|_2^{11}$$

$$= 2\pi \left[\left(\frac{2}{5}(11-2)^{5/2} + \frac{4}{3}(11-2)^{3/2} \right) - \left(\frac{2}{5}(2-2)^{5/2} - \frac{4}{3}(2-2)^{3/2} \right) \right]$$

$$= 2\pi \left[\frac{2}{5}(9)^{5/2} - 4/3 (9)^{3/2} - 0 \right]$$

② b) Revolve the region bounded by $y=x^2+2$, $x=0$, $x=3$, and $y=0$ around the x-axis. Find volume by shells.

same region as ①, but different axis of revolution.



The region above $y=2$ has height h of cylinder horizontal

right x — left x
($x=3$) (parabola) in y

The region below $y=2$ has height h of cylinder horizontal.

right x — left x
($x=3$) ($x=0$)

Because these regions have different expressions for height h, we need two integrals.

$$\int_{y=0}^{2} 2\pi\,\underbrace{(y-0)}_{\text{radius}}\,\underbrace{(3-0)}_{\text{height}}\,dy + \int_{y=2}^{11} 2\pi\,\underbrace{(y-0)}_{\text{same } r}\,\underbrace{(3-\sqrt{y-2})}_{\text{height}}\,dy$$

radius (variable of integration − axis of rev)
$y-0$

solve parabola for x-coord in terms of y

$y=x^2+2$

$y-2=x^2$

$\pm\sqrt{y-2}=x$

graph in QI $\Rightarrow$ choose $x=\sqrt{y-2}$

$$=2\pi\cdot 3\int_0^2 y\,dy + 2\pi\int_2^{11} y(3-\sqrt{y-2})\,dy$$

$u=y-2$ $\quad y=u+2$ $\quad u_1=2-2=0$
$du=dy$ $\quad u_2=11-2=9$

$$=6\pi\cdot\frac{y^2}{2}\Big|_0^2 + 2\pi\int_{u=0}^{9}(u+2)(3-u^{1/2})\,du$$

$$=3\pi(4-0) + 2\pi\int_{u=0}^{9} 3u-u^{3/2}+6-2u^{1/2}\,du$$

$$= 12\pi + 2\pi \left[\frac{3u^2}{2} - \frac{2}{5} u^{5/2} + 6u - 2 \cdot \frac{2}{3} u^{3/2} \right] \Bigg|_{0=u}^{9=u}$$

$$= 12\pi + 2\pi \left[\left(\frac{3}{2}(9)^2 - \frac{2}{5}(9)^{5/2} + 6(9) - \frac{4}{3}(9)^{3/2} \right) - 0 \right]$$

$$= 12\pi + 2\pi \left[\frac{243}{2} - \frac{2}{5}(3^5) + 54 - \frac{4}{3} \cdot (3)^3 \right]$$

$$= 12\pi + 2\pi \left[\frac{243}{2} - \frac{486}{5} + 54 - 36 \right]$$

$$= 12\pi + 2\pi \left[\frac{423}{10} \right]$$

$$= \boxed{\frac{486\pi}{5}} \quad \text{or} \quad \boxed{97.2\pi}$$

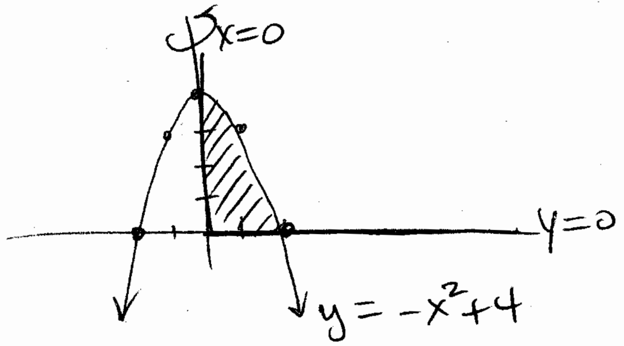
Math 250

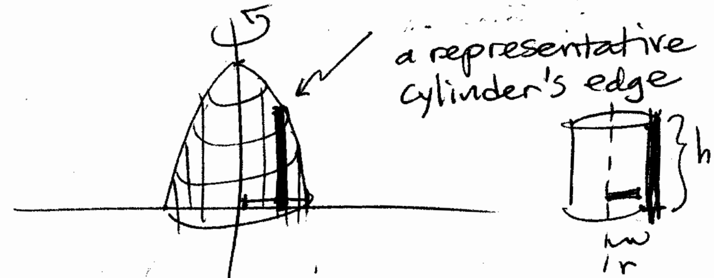
③ Find the volume of the solid of revolution formed by revolving the region bounded by

$$y = -x^2 + 4$$
$$x = 0$$
$$y = 0$$

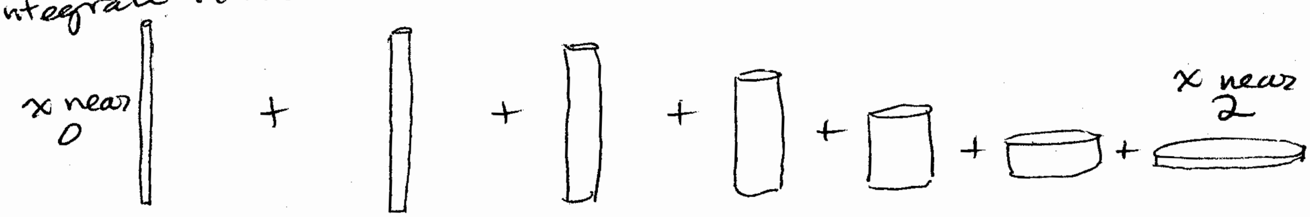
about the y-axis, using the method of cylindrical shells.

<u>step 1</u>: sketch the region and revolve it:





<u>step 2</u>: The cylindrical shells encircle the axis of revolution. make the radius of the cylinder perpendicular to the axis of revolution. Accumulate the surface areas of cylinders, for $x=0$ to $x=2$. $A = 2\pi \cdot r \cdot h$.
Integrate in radius..



The radius of each cylinder is simply x.
The height of each cylinder is $y = f(x)$, top-bottom.

<u>step 3</u>: Set up integral and evaluate

$$\int_{x=0}^{x=2} 2\pi\, r(x)\, h(x)\, dx = \int_0^2 2\pi \cdot x \cdot (-x^2+4)\, dx = \int_0^2 -x^3 + 4x\, dx$$

$$= 2\pi \left[-\frac{1}{4}x^4 + 2x^2 \right] \Big|_0^2 = 2\pi \left\{ [-4+8] - [0+0] \right\} = \boxed{8\pi}$$

Math 250

(1) Find the volume of the solid of revolution formed by revolving the region bounded by

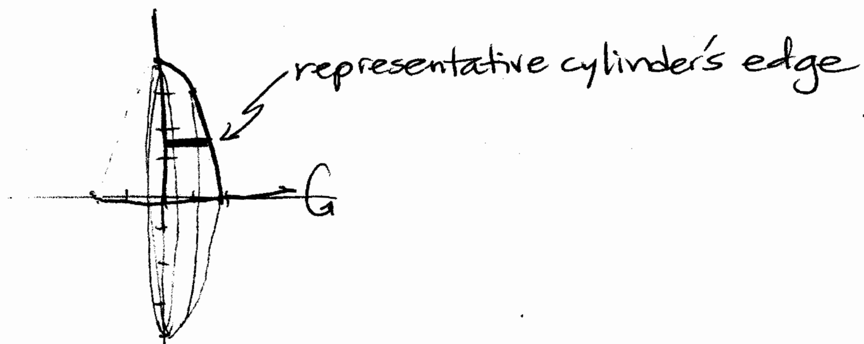
$y = -x^2 + 4$

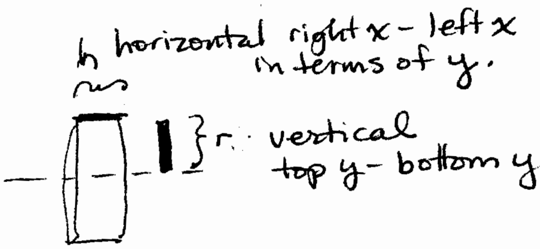
$y = 0$

$x = 0$

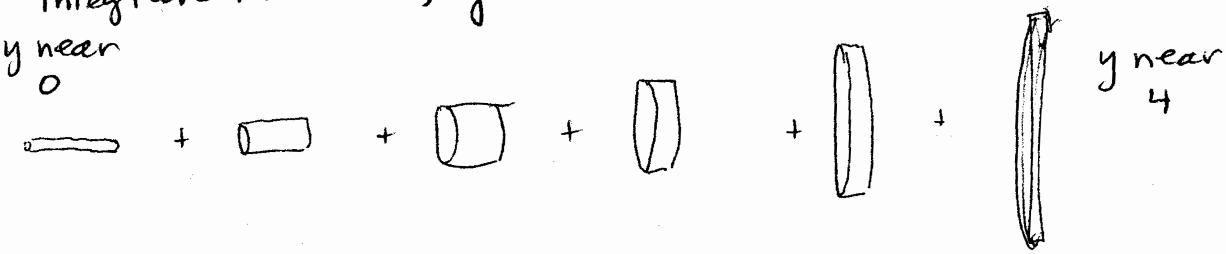
about the x-axis, using the method of cylindrical shells.

Step 1: sketch and revolve





step 2: The cylindrical shells encircle the axis of revolution so that the radius is perpendicular to the axis of revolution. Accumulate the surface areas $A = 2\pi r h$ for $y=0$ to $y=4$.
integrate in radius, $y \to 0$



The radius of each cylinder is simply y.
The height of each cylinder is $x = f^{-1}(y)$.

$$y = -x^2 + 4$$

$$y - 4 = -x^2$$

$$4 - y = x^2$$

$$\pm\sqrt{4-y} = x$$

$$\sqrt{4-y} = x$$ QI is (+) root.

step 3: Set up integral

$$\int_{y=0}^{y=4} 2\pi\, r(y)\cdot h(y)\, dy$$

need u-substitution to antidiff.

$$= \int_0^4 2\pi \cdot y \cdot \sqrt{4-y}\, dy$$

$u = 4-y$ $\qquad$ $y = 4-u$

$du = -dy$

$u_1 = 4-0 = 4$

$u_2 = 4-4 = 0$

$$= -2\pi \int_{u=4}^{u=0} (4-u)u^{1/2}\, du$$

$$= 2\pi \int_{u=0}^{u=4} 4u^{1/2} - u^{3/2}\, du$$

rewrite in u
simplify in u

$$= 2\pi \left[\frac{8}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right] \Big|_0^4$$

antidiff in u.

$$= 2\pi \left\{ \left[\frac{8}{3} \cdot (\sqrt{4})^3 - \frac{2}{5} (\sqrt{4})^5 \right] - 0 \right\}$$

evaluate in u.

{You could also evaluate in y, not shown here.}

$$= 2\pi \left\{ \frac{64}{3} - \frac{64}{5} \right\}$$

simplify

$$= 2\pi \cdot \frac{128}{15}$$

arithmetic

$$= \boxed{\frac{256\pi}{15}}$$

Math 250

(5) Find the volume of the solid of revolution formed by revolving the region bounded by

$y = x^2$

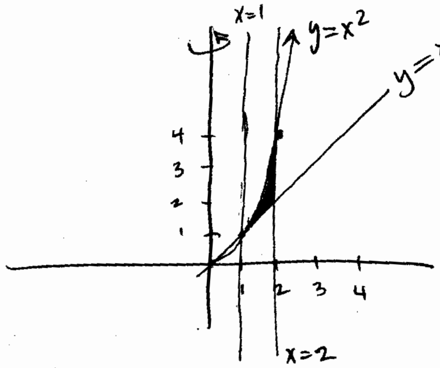
$x = 1$

$x = 2$

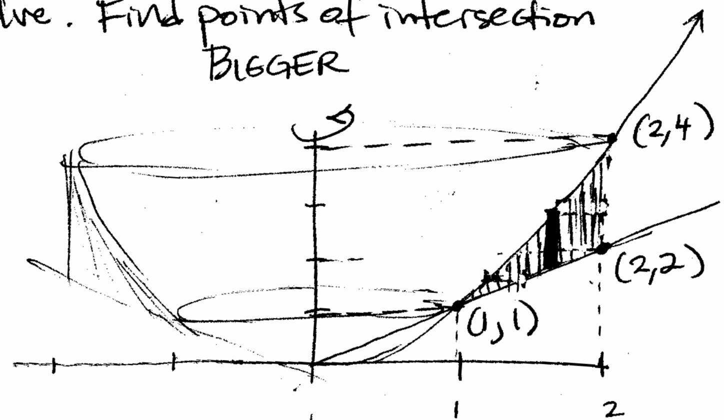
$y = x$

about the y-axis, using the method of cylindrical shells.

step 1: sketch and revolve. Find points of intersection



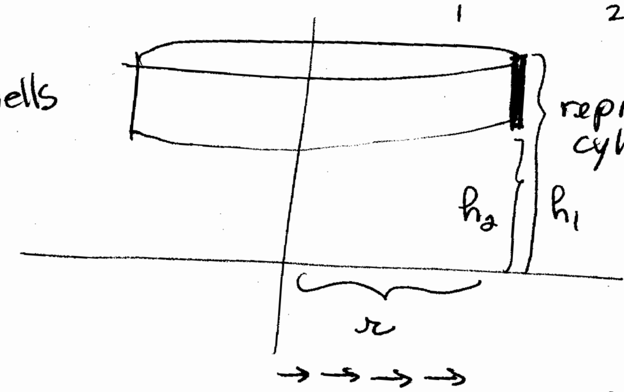
BIGGER



a bowl with a hole in the bottom and thick upperedges

step 2: the cylindrical shells encircle the axis of revolution, so that the radius is perpendicular to the axis of revolution

$2\pi r h$



integrate in x, radius

The radius is simply x.

The height that's desired requires subtracting away the lower part.

cylinder formed by parabola − cylinder formed by line

$$2\pi r h_1 \quad - \quad 2\pi r h_2$$

$$2\pi r (h_1 - h_2)$$

$h_1 = x^2$ height of parabola

$h_2 = x$ height of line.

step 3: Set up integral and evaluate.

$$\int_{x=1}^{x=2} 2\pi x (x^2 - x)\, dx$$

$$= 2\pi \int_1^2 x^3 - x^2 \, dx$$

$$= 2\pi \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 \right] \Big|_1^2$$

$$= 2\pi \left[\left(4 - \frac{8}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right) \right]$$

$$= 2\pi \left[\frac{4}{3} - \frac{-1}{12} \right]$$

$$= 2\pi \left(\frac{17}{12} \right)$$

$$= \boxed{\frac{17\pi}{6}}$$

Could this have been done by the disk/washer method?
Yes — but it would be
- integrated in y $\Rightarrow$ need to solve for x in terms of y
- require two integrals, plus subtraction.

$$\int_{y=1}^{y=4} \pi r^2 \, dy = \int_{y=1}^{y=4} \pi R^2 - \pi r^2 \, dy$$

$$\int_{y=1}^{y=2} \pi y^2 - \pi (\sqrt{y})^2 \, dy + \int_{y=2}^{y=4} \pi (2)^2 - \pi (\sqrt{y})^2 \, dy$$

$$= \pi \int_1^2 y^2 - y \, dy + \pi \int_2^4 4 - y \, dy$$

$$= \pi \left(\frac{1}{3}y^3 - \frac{1}{2}y^2 \right) \Big|_1^2 + \pi \left(4y - \frac{1}{2}y^2 \right) \Big|_2^4$$

$$= \pi \left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right) \right] + \pi \left[(16 - 8) - (8 - 2) \right]$$

$$= \pi \left[\frac{2}{3} + \frac{1}{6} + 8 - 6 \right] = \boxed{\frac{17\pi}{6}}$$

Math 250

⑥ Find the volume of the surface of revolution formed by revolving the region bounded by

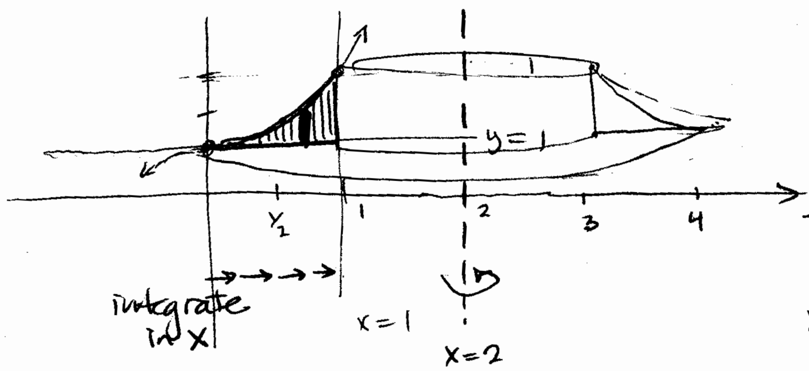
$y = x^3 + x + 1$

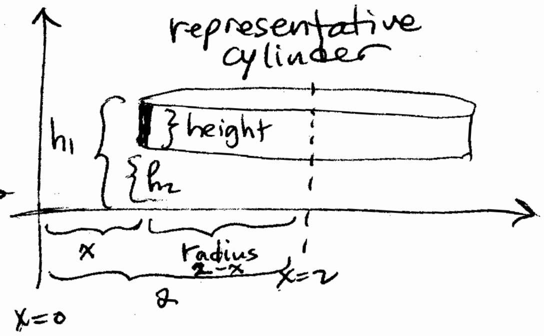
$y = 1$

$x = 1$

around $x = 2$.

<u>step 1:</u> Sketch and revolve — use GC!





<u>step 2:</u> The radius of the cylindrical shell is perpendicular to the axis of revolution. Area accumulated: $2\pi r h$

The radius is <u>NOT</u> x $\Rightarrow$ we don't want the distance from the y axis to the representative edge.

* We want the distance from $x = 2$ to the representative edge.

$$\underline{\underline{\text{radius} = 2 - x}}$$

The height is <u>NOT</u> y $\Rightarrow$ we don't want the distance from the x-axis to the top of the cylinder

* We want the distance from $y = 1$ to the top of the cylinder

$$\underline{\underline{\text{height} = h_1 - h_2 = (x^3 + x + 1) - 1 = x^3 + x}}$$

<u>step 3:</u> set up Integral

$$\int_{x=0}^{x=1} 2\pi r h \, dx = \int_0^1 2\pi(2-x)(x^3+x)\,dx$$

$$= 2\pi \int_0^1 2x^3 + 2x - x^4 - x^2 \, dx$$

$$= 2\pi \int_0^1 -x^4 + 2x^3 - x^2 + 2x \, dx$$

$$= 2\pi \left[-\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 - \tfrac{1}{3}x^3 + x^2 \right] \Big|_0^1$$

$$= 2\pi \left\{ \left(-\tfrac{1}{5} + \tfrac{1}{2} - \tfrac{1}{3} + 1 \right) - (0) \right]$$

$$= 2\pi \cdot \frac{29}{30}$$

$$= \boxed{\frac{29\pi}{15}}$$

Could this have been done by disks/washers?

- integrate in y
- need x as a function of y.

oops! $y = x^3 + x + 1$ can't be solved for x using skills we have.

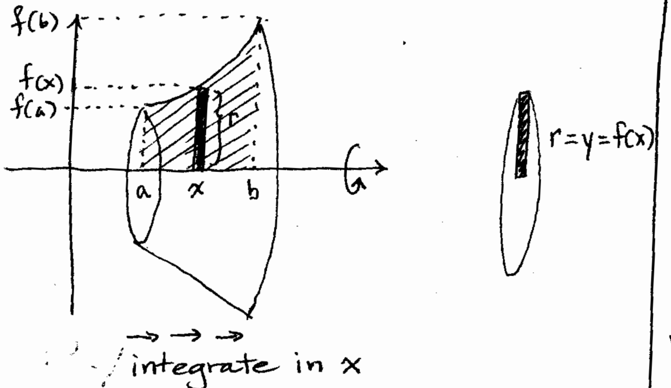
So no... we could <u>not</u> have done it with disks/washers.

Math 250

Let's compare and contrast the two methods and the two types of revolution:

* DISKS/WASHERS
revolve around x-axis

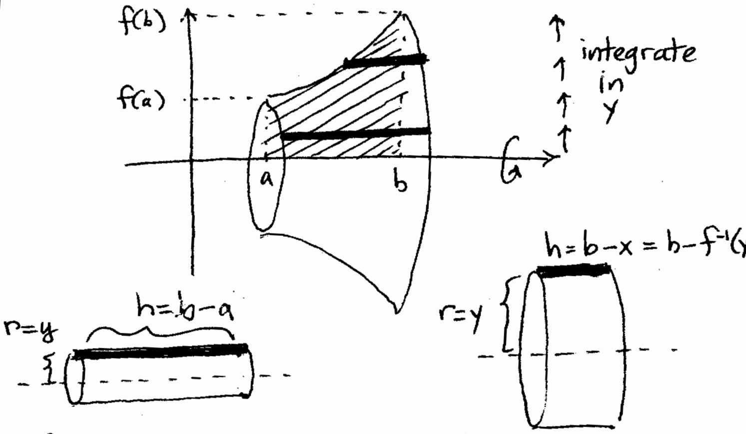
πr^2



$$\int_{x=a}^{x=b} \pi \left(f(x)\right)^2 \, dx$$

CYLINDRICAL SHELLS
revolve around x-axis

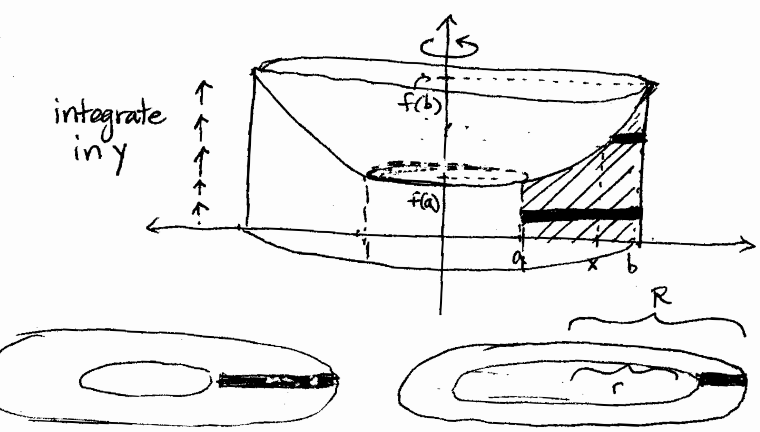
$2\pi r h$



$$\int_{y=0}^{y=f(a)} 2\pi y(b-a)\,dy + \int_{y=f(a)}^{y=f(b)} 2\pi y\left(b-f^{-1}(y)\right) dy$$

DISKS/WASHERS
revolve around y-axis

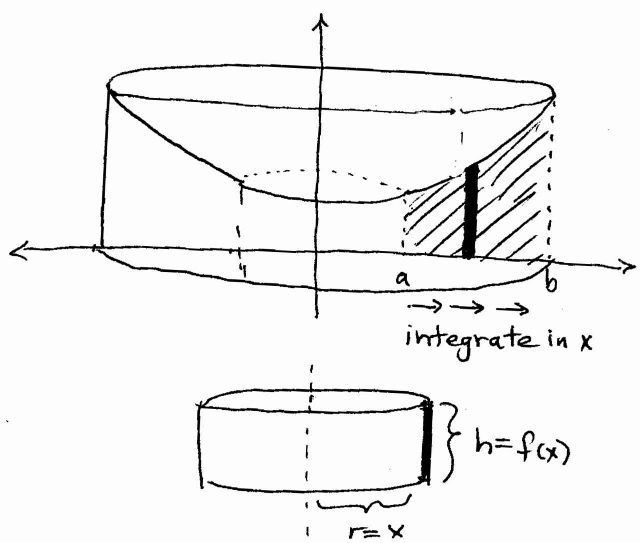
πr^2



$$\int_{y=0}^{y=f(a)} \pi b^2-\pi a^2\,dy + \int_{y=f(a)}^{y=f(b)} \pi b^2-\pi\left(f^{-1}(y)\right)^2 dy$$

* CYLINDRICAL SHELLS
revolve around y-axis

$2\pi r h$



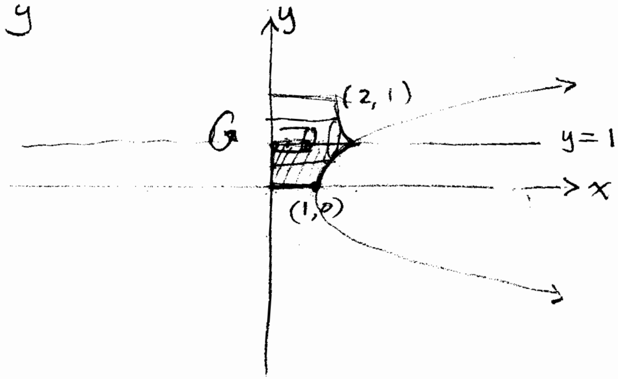
$$\int_{x=a}^{x=b} 2\pi x\cdot f(x)\,dx$$

Math 250

7 EX 6.3 & 6.4

$$\begin{cases} x = y^2+1 \\ x = 0 \\ y = 0 \\ y = 1 \end{cases} \Rightarrow \pm\sqrt{x-1} = y$$

around $y=1$



By shells

$$\int_{y=0}^{1} 2\pi \overbrace{(1-y)}^{r(y)}\overbrace{(y^2+1-0)}^{h(y)}\,dy +$$

By slices

disks! no space between shaded region and axis of revolution

$$\int_{x=0}^{1} \overbrace{\pi 1^2}^{\text{disk } \pi r^2} dx + \int_{x=1}^{2} \pi(1-\sqrt{x-1})^2\,dx$$

$\boxed{\text{shells}}$

$$2\pi\int_{y=0}^{1}(1-y)(y^2+1)\,dy = 2\pi\int_0^1 y^2+1-y^3-y\,dy = 2\pi\int_0^1 -y^3+y^2-y+1\,dy$$

$$= 2\pi\left[-\tfrac{1}{4}y^4+\tfrac{1}{3}y^3-\tfrac{1}{2}y^2+y\right]\Big|_0^1 = 2\pi\left[\left(-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{2}+1\right)-0\right] = \boxed{\frac{7\pi}{6}}$$

$\boxed{\text{slices}}$

$$\pi\int_{x=0}^{1}dx + \pi\int_{x=1}^{2} 1-2\sqrt{x-1}+x-1\,dx = \pi x\Big|_0^1 + \pi\int_1^2 x\,dx - 2\pi\int_1^2\sqrt{x-1}\,dx$$

$u = x-1 \quad u_1 = 1-1 = 0$

$du = dx \quad u_2 = 2-1 = 1$

$$= \pi(1-0) + \pi\left(\tfrac{1}{2}x^2\right)\Big|_1^2 - 2\pi\int_0^1 u^{1/2}\,du = \pi + \pi\left(\tfrac{1}{2}\cdot 2^2-\tfrac{1}{2}1^2\right) - 2\pi\left[\tfrac{2}{3}u^{3/2}\right]\Big|_0^1$$

$$= \pi + \pi\left(2-\tfrac{1}{2}\right) - \tfrac{4\pi}{3}\left(1^{3/2}-0^{3/2}\right) = \pi + \tfrac{3}{2}\pi - \tfrac{4\pi}{3} = \boxed{\frac{7\pi}{6}}$$